Tides – what's really going on?

I never quite got tides when I was at school. The part about the sea being pulled upwards on the side towards the Moon was OK. But we were told the sea also bulges out on the far side too – although this was glossed over very quickly. Immediately diagrams including the Sun were produced, and we moved onto springs and neaps before anyone had time to ask awkward questions.

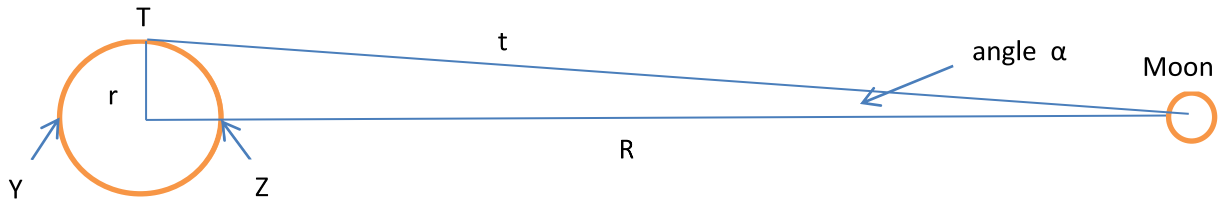
I recently sought enlightenment on the internet. Explanations were generally over-simplified and maths-averse. Of course, there were some rigorous treatments, but then the maths quickly got scary.

I thought I'd see if I could answer the following three questions convincingly:

- Why is there a bulge on the far side too?
- Why is it always shown to be the same size as the bulge on the moon side?
- How high are the bulges expected to be?

This is what I came up with, keeping the maths as simple as I could. Acknowledgements are due to Sir Isaac Newton for his laws of mechanics and gravitation, and his binomial theorem. I didn't need to use his calculus.

<u>The underlying mechanism</u>



Earth, on left, has radius r, and its centre is a distance R from the centre of the Moon.

Newton's Law of Gravitation expresses the attracting gravitational force between two masses. It is proportional to the mass of the body, or small part of a body, which is under consideration. Given Newton's Third Law of Motion, force = mass x acceleration, it is often useful to think of the attraction as a gravitational acceleration, even if there is no obvious motion. So, the force I exert on the floor, my weight, is my mass times the 'acceleration due to gravity', even though the reaction force from the floor means that I experience no net force, and do not accelerate anywhere.

All points on and inside the Earth experience the gravitational pull of the Moon. The variation of the gravitational acceleration from the Moon at different points on the Earth's surface causes tides. We will take the reference point for this variation as the Earth's centre of mass. (This seems reasonable, based on symmetry, but it will need some further justification, which I'll give later.)

At the Earth's centre the gravitational acceleration is $\frac{GM}{R^2}$, where G is the gravitational constant and M is the mass of the Moon. Consider the acceleration at two points on the Earth/Moon axis, each at the Earth's surface at a distance r from its centre. The acceleration at the point closer to the Moon, Z, is $\frac{GM}{(R-r)^2}$, and at the farther point, Y, it is $\frac{GM}{(R+r)^2}$

The *tidal* acceleration at any point is the difference from the acceleration at the centre of the Earth. At Z it is therefore $\frac{GM}{(R-r)^2} - \frac{GM}{R^2}$, in the direction away from the centre of the Earth, towards the Moon. This rearranges to

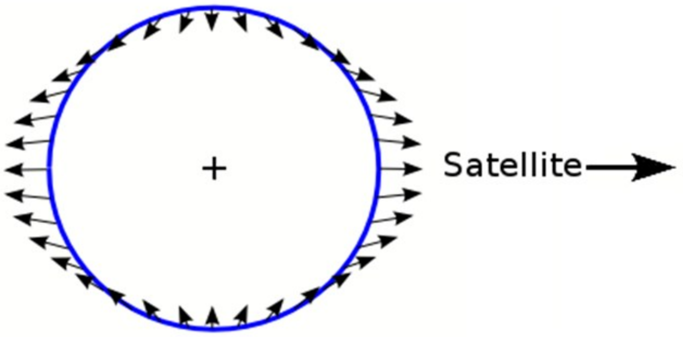
$\frac{GM}{R^2}\left(\frac{1}{\left(1-\frac{r}{R}\right)^2}-1\right)$. By a similar argument, the far-side tidal acceleration at Y, away from the centre of the Earth, but this time *away* from the Moon, is $\frac{GM}{R^2}\left(1-\frac{1}{\left(1+\frac{r}{R}\right)^2}\right)$.

This shows that the tidal effect is not exactly the same size on each side of the Earth. However, we know r is much smaller than R. Expand both results using the binomial theorem, and ignore the higher order terms involving $\frac{r}{R}$. The result in each case is $\frac{GM}{R^2}.2\left(\frac{r}{R}\right)=\frac{2GMr}{R^3}$. For most practical purposes the effect on each side is therefore the same. This goes a long way towards explaining the first two of my questions.

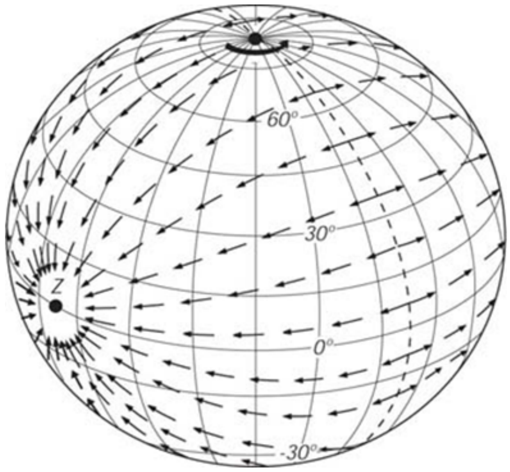
Note how the effect drops as the cube of the distance to the tide-inducing body. This is how the tidal effect of the Sun comes to be rather less than that of the Moon.

Now consider point T in the diagram, at a distance t from centre of the Moon. The gravitational acceleration owing to the Moon has a component downwards, perpendicular to the Earth-Moon axis. This is wholly tidal, as it is entirely absent at the centre of the Earth. The downwards component is $\frac{GM}{t^2}.\sin\alpha$, which is $\frac{GMr}{t^3}$. The distance t can be worked out using Pythagoras, but when higher order terms in $\frac{r}{R}$ are ignored, t is the same as R. For most practical purposes, the downwards component can be taken as $\frac{GMr}{R^3}$. This is half the size of the effect at points Y and Z.

The tidal effects all the way round can be found from a fuller analysis, as in http://oceanworld.tamu.edu/resources/ocng_textbook/chapter17/chapter17_04.htm. But enough work has been done to make the picture below plausible.



It is actually horizontal forces in the ocean which create the tides, and it can be seen that these will be strongest in the mid-latitudes. The picture below shows the horizontal components of the tidal forces across the globe. If the Earth was entirely covered in water, this can be thought of as the way the tidal bulges are kept propped up.



The Moon is above point Z. Note that Z will not generally be on the equator. It can be anywhere in the tropics.

In summary, tides arise because the Earth is within the Moon's "gravity gradient" - the way the gravity from the Moon varies in space.

Height of the tide

What sort of value does this approach imply for the size of the bulge? The interest is in h, the difference in the heights of the ocean at Z (or Y) and at T. For a unit test mass, this would normally represent a difference in potential energy of gh, where g is the acceleration due to gravity at the surface of the Earth. But the potential at T and Z must be the same, otherwise the water would start to flow in order to redistribute itself. Somehow the tidal effect must have contributed an extra gh in potential at T.

Imagine taking the unit test mass from the centre of the Earth, where the tidal acceleration is zero, to point Z. The average assisting force from the tidal effect would be $\frac{1}{2} \cdot \frac{2GMr}{R^3}$, operating over a distance r. The work supplied to you would be $\frac{GMr^2}{R^3}$. To take the unit mass from the centre of the Earth to T, the tidal effect would now resist you, and you would have to do extra work on it of $\frac{1}{2} \cdot \frac{GMr^2}{R^3}$. So the potential energy of the unit mass at T is $\frac{3GMr^2}{2R^3}$ greater than that at Z.

We can equate this to gh. Also, by considering the gravitational attraction of the Earth, of mass E, on a test mass at the surface, we know that $g = \frac{GE}{r^2}$.

This leads to $h = \frac{3}{2} \cdot \left(\frac{M}{E}\right) \cdot \left(\frac{r}{R}\right)^3 \cdot r$.

The Earth has about 80 times the mass of the Moon, so $\left(\frac{M}{E}\right)$ is 1/80. The Moon is about 60 Earth radii away, so $\left(\frac{r}{R}\right)$ is 1/60. The radius of the Earth is 6370 km. This implies h = 0.55 metres. This is not very large, but it is not far off the tidal range observed at many mid-ocean locations. It needs a lot of interpretation before it can be related to marine tides in general.

Doing the same sum for the Sun's tidal effect on the Earth gives h = 0.25 metres.

Consideration of the barycentre

The story so far has conveniently glossed over why the gravitational attraction doesn't simply make the Earth and the Moon crash into each other. The answer lies in the Moon's revolution round the Earth.

Strictly the Moon does not revolve around the Earth, but the whole system rotates around the 'barycentre', the centre of mass of the Earth-Moon system. The barycentre actually sits inside the Earth itself, about 1000 miles below the surface. The classic picture is of a highly asymmetrical dumbbell rotating in space, with the Earth doing a kind of wobble. Because of this motion, the centres of both the Earth and the Moon accelerate towards the barycentre, in just the way which matches their gravitational accelerations.

The dumbbell picture is a little bit misleading, and needs some refinement. It has the Earth rotating on an axis through its centre, in a way that does not actually happen.

(Of course, the Earth does rotate about an axis, but much faster than this, once every day. The effect is considerable, but quite separate. It squashes the Earth, so that the polar diameter is 42 km less than the equatorial diameter. But as it acts uniformly around the planet, and does not change with time, we don't really notice it.)

Allow the Earth-Moon system to rotate about the barycentre, but unlike the dumbbell case do not allow the Earth to rotate about an axis through its centre. The barycentre can be taken as a fixed point in space, but it will not be a point fixed within the Earth. The Earth's centre will move in a circle, as before. The radius of the circle will be the distance from the barycentre to the centre of the Earth. This is not easy to picture, but consider the motion of every other point inside and on the Earth. Each point will move at the same speed as the Earth's centre, and in the same direction, describing a circle of the same radius. The centre of each circle will be fixed in space, and at any given time the point itself will be displaced from it in the direction parallel to the Earth-Moon line. Every point has a similar motion, and exactly the same acceleration, in both magnitude and direction, as the acceleration at the centre of the Earth. This is accounted for by the Moon's gravitational attraction at Earth's centre of mass, and is the $\frac{GM}{R^2}$ term seen earlier, acting parallel to the Earth-Moon line.

So, the use of the centre of the Earth as the reference for all the tidal accelerations can be justified.

<u>Tides in practice</u>

In principle, the tidal bulges would stay put opposite the Moon while the Earth rotates on its own axis underneath. Because the Earth is tilted on its axis, the bulges would also sweep back and forth, north and south, across the tropics, as the Moon progresses in its orbit. On average, the Moon reaches its highest point over a given spot every 24.84 hours, so a bulge should go past every 12 hours and 25 minutes. (The smaller bulges caused by the Sun should pass every 12 hours.)

But seawater is viscous, and the oceans are not deep enough to allow disturbances to travel so quickly as that. In any case, the continents and the ocean basins interfere. In shallow bays and estuaries, local resonances are set up and have a considerable effect. It all becomes very complicated.

Nevertheless, in many places the dominating frequency of the tide is once every 12 hours 25 minutes.